A Unified Framework for Defining and Identifying Causal Effects

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Introduction

Causal concerns were central to
- Early development of IV

- Cowles Commission development of simultaneous equations

Lack of clear meaning of cause and effect led to a decline in attention to causal issues

A resurgence in attention to causality began in 70's and 80's
- Labor economics literature
- Epidemiology/treatment effects literature
- Machine learning literature

Current approaches to causality are not fully compatible

Goal of current research:

Provide unified framework accommodating previous approaches

- Yields rigorous notions of cause and effect
- Delivers conditions ensuring identification of effects of interest
- Suggests statistical/econometric methods for estimating causal effects

Additional Benefits

- Generalizes notion of exogeneity (conditional exogeneity)
- Clarifies interpretation of regression coefficients and their estimates
- Relaxes SUTVA of treatment effects literature
- Extends machine learning framework to permit mutual causality
- Provides insight into selection of covariates
- Delivers tests for identification of causal effects
- Provides basis for extending concept of instrumental variables

SETTABLE SYSTEMS

Definition 2.1: Let $(\Omega, \mathcal{F}, P)$ be a complete probability space, and let $\mathcal{A}$ be a non-empty multi-dimensional Borel set. For *agents* $h = 1, 2, \ldots$, let *attributes* a_h belong to $\mathcal{A}$, and put $a \equiv \{a_h\}$. For $h = 0, 1, \ldots,$ and $j = 1, 2, \ldots,$ let *settings* $\mathcal{Z}_{h,j} : \Omega \to \mathcal{R}$ be measurable functions. For $h = 1, 2, \ldots$ and $j = 1, 2, \ldots$, let *attribute-indexed* response functions $r_{h,j} : \mathcal{R}^\infty \times \mathcal{A}^\infty \to \mathcal{R}$ be measurable functions.

For $h = 0$ and $j = 1, 2, \ldots$, let $Z_{(h,j)}$ be the vector including every seting except $Z_{h,j}$, and define the *attribute-indexed settable variables* $X_{h,j} : \{0, 1\} \times \Omega \to R$ such that

$$\begin{aligned} \mathcal{X}_{h,j}(1, \cdot) &= Z_{h,j} \\ \mathcal{X}_{h,j}(0, \cdot) &= r_{h,j}(Z_{(h,j)}, a). \end{aligned}$$

For $h = 0$ and $j = 1, 2, \ldots,$ let $\mathcal{X}_{h,j}(0, \cdot) = \mathcal{X}_{h,j}(1, \cdot) = Z_{h,j}$.

Put $Z \equiv \{Z_{h,j} : j = 1, 2, \ldots; h = 0, 1, \ldots\}$, $r \equiv \{r_{h,j}, j = 1, 2, \ldots; h = 1, 2, \ldots\}$, and $\mathcal{X} \equiv \{X_{h,j}, j = 1, 2, \ldots; h = 0, 1, \ldots\}$.

The pair $\mathcal{S} \equiv \{(\Omega, \mathcal{F}, P), (\mathcal{A}, a, Z, r, \mathcal{X})\}$ is an *attribute-indexed settable system.* ■

CAST OF CHARACTERS

$(\Omega, \mathcal{F}, P)$ Probability space
governs all randomness

$\mathcal{A}$ Attribute space
describes agent characteristics

$a_h \in \mathcal{A}$ Agent characteristics; agent $h = 1, 2, \ldots$

$Z_{h,j}$ $h = 0, 1, 2, \ldots;$ $j = 1, 2, \ldots$
real-valued random variables
settings

$r_{h,j}$ $h = 1, 2, \ldots;$ $j = 1, 2, \ldots$
response functions

SETTABLE VARIABLES

$h = 1, 2, \ldots;\ j = 1, 2, \ldots$

$$X_{h,j}(1,\cdot) = Z_{h,j}$$

$X_{h,j}$ is set to $Z_{h,j}$

$$X_{h,j}(0,\cdot) = r_{h,j}(Z_{(h,j)}, a)$$

$X_{h,j}$ is free to respond

- Response is determined by agent h
- Depends on settings $Z_{(h,j)}$ of all other settable variables
- Depends on characteristics of all agents $a = \{a_1, a_2, \ldots\}$

SETTABLE VARIABLES (continued)

$h = 0;\ j = 1, 2, \ldots$

$$x_{0,j}(1,\cdot) = Z_{0,j}$$

Convention:

$$x_{0,j}(0,\cdot) = Z_{0,j}$$

fundamental settings

Not determined by any other variables of the systems, e.g.,

- initial values
- default values

SETTABLE SYSTEM (continued)

$$\mathcal{S} = \{(\Omega, \mathcal{F}, P), (\mathcal{A}, a, Z, r, \mathcal{X})\}$$

$$Z = \{Z_{h,j}, h = 0,1,2,\ldots; j = 1,2,\ldots\}$$

$$r = \{r_{h,j}, h = 1,2,\ldots; j = 1,2,\ldots\}$$

$$\mathcal{X} = \{\mathcal{X}_{h,j}, h = 0,1,2,\ldots; j = 1,2,\ldots\}$$

Stochastic structure

$$(\Omega, \mathcal{F}, P)$$

Economic structure/causal structure

$$(\mathcal{A}, a, Z, r, \mathcal{X})$$

CAUSALITY

Cause and effect defined using response functions

Key Idea: Suppose

$$r_{h,j}(z_{(h,j)}, a)$$

is constant in $z_{i,k}$

for all values of other elements of $z_{(h,j)}$

Then $x_{i,k}$ *does not cause* $x_{h,j}$

$$x_{i,k} \Rightarrow\!|_{S}\; x_{h,j}$$

Otherwise $x_{i,k} \Rightarrow_{S} x_{h,j}$

EFFECTS

Marginal ceteris paribus effect on $x_{h,j}$ of $x_{i,k}$

$$\frac{\partial r_{h,j}}{\partial z_{i,k}}(z_{(h,j)}, a)$$

Effect on $x_{h,j}$ of intervention $z_{(h,j)} \to z^*_{(h,j)}$ to $x_{(h,j)}$

$$\begin{aligned}\Delta r_{h,j}(z_{(h,j)}, z^*_{(h,j)}, a) \\ \equiv r_{h,j}(z^*_{(h,j)}, a) - r_{h,j}(z_{(h,j)}, a)\end{aligned}$$

INTERVENTIONS

Conceptual operation

$$z_{(h,j)} = Z_{(h,j)}(\omega) \quad \omega \in \Omega$$
$$z^*_{(h,j)} = Z_{(h,j)}(\omega^*) \quad \omega^* \in \Omega$$

A pair $(\omega, \omega^*) \in \Omega \times \Omega$ is an *intervention*

Either ω or ω^* or both may be *counterfactual*

Attributes a are not subject to intervention

- cannot have effects
- act to modify responses and effects

FORMAL CAUSALITY

Definition 2.2 Let (Ω, F, P), A , a, and Z be as in Definition 2.1. Suppose that settings of $X_{h,j}$ are given by $X_{h,j}$ (1, ·) = Z_{hj} , j = 1, 2, . . . , h = 0, 1, . . . , and let $\Pi = \{\Pi_b\}$ be a partition of the ordered pairs {(h, j): j = 1, 2,. . .; h = 1, 2, . . . }. Suppose there exists a countable sequence of measurable functions $r^{\Pi} \equiv \{r^{\Pi}_{h,j}\}$ such that for all (h, j) in Π_b the responses $Y_{h,j}$ = $X_{h,j}$ (0, ·) are jointly determined as

$$Y_{h,j} = r^{\Pi}_{h,j}(Z_{(b)}, a), \qquad b = 1, 2, \ldots,$$

where $Z_{(b)}$ is the countable vector containing $Z_{i,k}$, $(i, k) \notin \Pi_b$. Then $S \equiv \{(\Omega, F, P), (A, a, Z, \Pi, r^{\Pi}, X)\}$ is an attribute-indexed partitioned settable system. ■

NEW STRUCTURE:

Partition $\Pi = \{\Pi_b\}$

- Π_b groups together response for all (h, j) in Π_b

- Responses depend only on $z_{(b)}$ (settings outside the group)

- r^{Π} indicates response function depends on partition

EXAMPLES

$\Pi_b = \{(h, j)\}$ elementary partition

$\Pi_b = \{(b, j), j = 1,2,\ldots\}$

- All responses governed by agent b.

- Permits joint responses for agent b to represent agent's joint optimal response given settings for all other agents

$\Pi_b = \{(h, j), h \in H_b, j = 1,2,\ldots\}$

- All responses governed by firm in industry H_b.

- Permits joint responses to represent industry equilibrium

Notation: $z_{(b)(i,k)}$ denotes all variables not in block b except (i, k)

DEFINITION 2.3 Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}, P), (\mathcal{A}, a, Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be an attribute-indexed partitioned settable system. For given positive integer b, let $(h, j) \in \Pi_b$. (i) If for given $(i,k) \notin \Pi_b$ the function $z_{i,k} \to r^{\Pi}_{h,j}(z_{(b)}, a)$ is a constant in $z_{i,k}$ for every $z_{(b),(i,k)}$, then we say $\mathcal{X}_{i,k}$ *does not cause* $\mathcal{X}_{h,j}$ *in* $\mathcal{S}$ and write $\mathcal{X}_{i,k} \not\Rightarrow_{\mathcal{S}} \mathcal{X}_{h,j}$. Otherwise, we say $\mathcal{X}_{i,k}$ *causes* $\mathcal{X}_{h,j}$ in $\mathcal{S}$ and write $\mathcal{X}_{i,k} \Rightarrow_{\mathcal{S}} \mathcal{X}_{h,j}$. (ii) For $(i,k),(h,j) \in \Pi_b$, $\mathcal{X}_{i,k} \not\Rightarrow_{\mathcal{S}} \mathcal{X}_{h,j}$. ■

Key Idea: If

$$r^{\Pi}_{h,j}(z_{(b)}, a)$$

is constant as a function of $z_{i,k}$, then

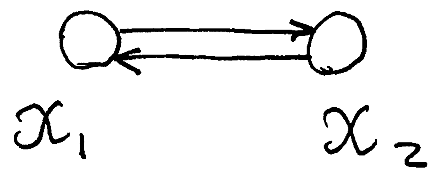
$$x_{i,k} \Rightarrow\!|_{\mathcal{S}}\ x_{h,j}$$

Otherwise

$$x_{i,k} \Rightarrow_{\mathcal{S}} x_{h,j}$$

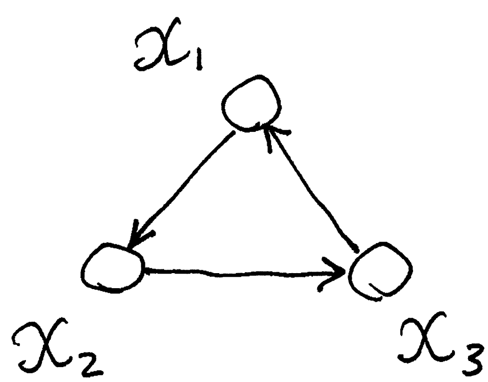
Part (ii) rules out causality within blocks

MUTUAL CAUSALITY



Permitted so far

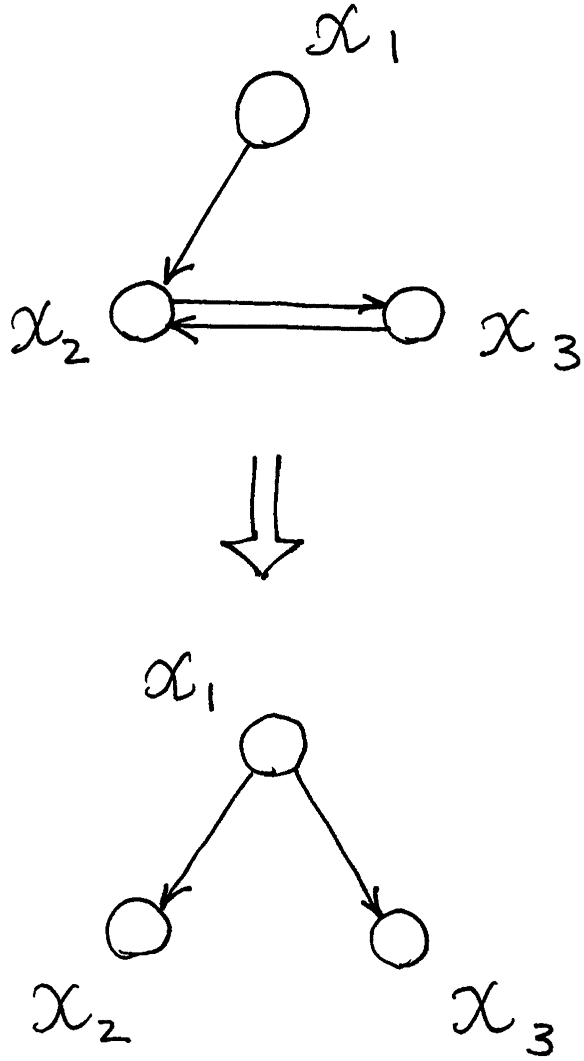
CAUSAL CYCLES



Permitted so far

- Analogous to simultaneity
- Complex to analyze

Partitioning can deliver recursive causal structures



Straightforward to analyze

RECURSIVE SETTABLE SYSTEMS

DEFINITION 2.4 Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}, P), (\mathcal{A}, a, Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be an attribute-indexed partitioned settable system. For b = 1, 2, …, let $\mathcal{X}_{h,j}$ for $(h, j) \in \Pi_b$ and let $\mathcal{X}_{[0]} = \mathcal{X}_0$. If Π is such that $\mathcal{X}_{[b]} \Rightarrow|_{\mathcal{S}} \mathcal{X}_{[0]},\ldots,\mathcal{X}_{[b-1]}$, b = 1, 2, …, then $\mathcal{S}$ is an *attribute-indexed recursive settable system*. ■

Higher level blocks *succeed* lower level blocks

Lower level blocks *precede* high level blocks
Successors do not cause predecessors

Predecessors may cause successors

If $\mathcal{X}_{i,k}$ precedes $\mathcal{X}_{h,j}$ write

$$\mathcal{X}_{h,j} \Leftarrow_{\mathcal{S}} \mathcal{X}_{i,k}$$

DEFINITION 2.5 Let $\mathcal{S} \equiv \{(\Omega, \mathcal{F}, P),$ $(\mathcal{A}, a, Z, \Pi, r^{\Pi}, \mathcal{X})\}$ be an attribute-indexed recursive settable system. Suppose the settings are *canonical settings* such that

$$\mathcal{X}_{[b]}(1,\cdot) = Z_{[b]} = r^{\Pi}_{[b]}(\mathcal{X}_{[0]}(1,\cdot), \ldots, \mathcal{X}_{[b-1]}(1,\cdot)),$$

$b = 1, 2, \ldots.$ Then $\mathcal{S}$ is an *attribute-indexed recursive settable system.* ■

Responses at level b are settings for all successors

Canonical settings are those generated in the absence of experimental control

Recursivity eliminates mutual causality, cycles

It does not eliminate "endogeneity"

SETTABLE SYSTEMS GENERATING SAMPLES

Applications typically focus on a collection of similar agents (firms, individuals, markets)

$\mathbf{A} \subset \mathcal{A}$ identifies population of interest

Response of interest for $a_h \in \mathbf{A}$

$$Y_h \stackrel{c}{=} r(Z_h, a_h, a_{(h)})$$

$$\mathcal{Y} \Leftarrow_S \mathcal{Z}$$

Sample:

Random integers $H_i \quad i = 1,2,\ldots.$

$$a_{H_i} \in \mathbf{A}$$

$$Y_{H_i} \stackrel{c}{=} r(Z_{H_i}, a_{H_i}, a_{(H_i)})$$

or

$$Y_i \stackrel{c}{=} r(Z_i, A_i)$$

for convenience

$\mathcal{S}$ generates a sample from **A** *involving* $(\mathcal{Y}, \mathcal{Z})$

Assumption B.1 Let an attribute-indexed recursive settable system $\mathcal{S}$ generate a sample from $\mathbf{A} \subset \mathcal{A}$ involving settable variables $(\mathcal{Y}, \mathcal{D}, \mathcal{W}, \mathcal{Z})$ such that $\mathcal{Y} \Leftarrow_{\mathcal{S}} (\mathcal{D}, \mathcal{W}, \mathcal{Z})$. In addition:

(a) Let attributes $\{(A_i, \tilde{B}_i) \equiv (\tilde{A}_i, \ddot{A}_i, \tilde{B}_i)\}$ be a sequence of random vectors, and let $(\mathcal{D}, \mathcal{W}, \mathcal{Z})$ generate settings $\{(D_i, W_i, Z_i) \equiv (D_i, W_i, \tilde{Z}_i, \ddot{Z}_i)\}$ such that the joint distribution of $(D_i, X_i) \equiv (D_i, W_i, \tilde{Z}_i, \tilde{A}_i, \tilde{B}_i)$ is H and the conditional distribution of $\ddot{X}_i \equiv (\ddot{Z}_i, \ddot{A}_i)$ given $(D_i, X_i) = (d, x)$ is $G(\cdot \mid d, x)$ for all $i = 1, 2, \ldots$, where D_i is $\mathbb{R}^{k_1}$-valued, k_1 a positive integer, W_i is $\mathbb{R}^{k_2}$-valued, $k_2 \in \mathbb{N}$, $\tilde{Z}_i$ is $\mathbb{R}^{k_3}$-valued, $k_3 \in \mathbf{N}$, $\ddot{Z}_i$ is $\mathbb{R}^{\infty}$-valued, $\tilde{A}_i$ is ℓ_1-valued, $\ell_1 \in \mathbb{N}$, $\ddot{A}_i$ is $\mathbb{R}^{\infty}$-valued, and $\tilde{B}_i$ is ℓ_2-valued, $\ell_2 \in \mathbb{N}$;

(b) The responses $\{Y_i\}$ of $\mathcal{Y}$ are determined as

$$Y_i \overset{c}{=} r(D_i, Z_i, A_i), \qquad i = 1, 2, \ldots,$$

where r is an unknown measurable scalar-valued function;

(c) (i) $\mathcal{D} \Leftarrow_{\mathcal{S}} (\mathcal{W}, \mathcal{Z})$; (ii) $\mathcal{W} \Leftarrow_{\mathcal{S}} \mathcal{Z}$

(d) The realizations of $Y_i, D_i, W_i, \tilde{Z}_i, \tilde{A}_i$ and $\tilde{B}_i$ are observed; those of $\ddot{Z}_i$ and $\ddot{A}_i$ are not. ■

CAST OF CHARACTERS

Settable Variables

$\mathcal{Y}, \mathcal{D}, \mathcal{W}, \mathcal{Z}$

$$\mathcal{Y} \Leftarrow_S (\mathcal{D}, \mathcal{W}, \mathcal{Z})$$

$$\mathcal{D} \Leftarrow_S (\mathcal{W}, \mathcal{Z})$$

$$\mathcal{W} \Leftarrow_S \mathcal{Z}$$

Observe $Y_i, D_i, W_i, \widetilde{Z}_i$

Don't observe $\ddot{Z}_i$, $Z_i = (\widetilde{Z}_i, \ddot{Z}_i)$

Attributes

Observe $\widetilde{A}_i, \widetilde{B}_i$

Don't observe $\ddot{A}_i$, $A_i = (\widetilde{A}_i, \ddot{A}_i)$

RESPONSE OF INTEREST

$$Y_i \stackrel{c}{=} r(D_i, Z_i, A_i) \quad i = 1,2,\ldots$$

$$= r(D_i, \widetilde{Z}_i, \ddot{Z}_i, \widetilde{A}_i, \ddot{A}_i)$$

Roles:

D_i causes of interest

Z_i auxiliary causes

A_i relevant response/effect modifiers

Note:

$W_i, \widetilde{B}_i$ don't appear - *structurally irrelevant*

COVARIATE-CONDITIONED AVERAGE EFFECTS

Average counterfactual response

$$\rho(d,x) \equiv E(r(d,Z,A) \mid X = x)$$
$$= \int r(d,\tilde{x},\ddot{x})\, dG(\ddot{x} \mid x)$$

$$\tilde{x} = (\tilde{z},\tilde{a})$$

$$\ddot{x} = (\ddot{z},\ddot{a})$$

$$x = (w,\tilde{z},\tilde{a},\tilde{b})$$

Average marginal effect

$$\frac{\partial \rho(d,x)}{\partial d_j} = \int \frac{\partial r}{\partial d_j}(d,\tilde{x},\ddot{x})\, dG(\ddot{x} \mid x)$$

Standard Conditional Expectation

$$\mu(d,x) \equiv E(r(D,Z,A) \mid D = d, X = x)$$
$$= \int r(d,\tilde{x},\ddot{x})dG(\ddot{x} \mid d,x)$$

$$\frac{\partial \mu(d,x)}{\partial d_j} = \int \frac{\partial r}{\partial d_j}(d,\tilde{x},\ddot{x})\,dG(\ddot{x} \mid d,x)$$
$$+ \int r(d,\tilde{x},\ddot{x}) \frac{\partial}{\partial d_j} dG(\ddot{x} \mid d,x)$$

Generally

$$\frac{\partial \mu(d,x)}{\partial d_j} \neq \frac{\partial \rho(d,x)}{\partial d_j}$$

Key condition:

Conditional Exogeneity

$$\ddot{X} \perp D \mid X$$

Assumption B.2 $\ddot{X} \perp D \mid X$. ■

Theorem 4.1 Suppose assumption B.1(a,b) hold and that $E(Y) < \infty$, (i) Then $\mu(D, X) \equiv E(Y \mid D, X)$ exists and is finite, and for each (d, x) in supp (D, X)

$$\mu(d, x) = \int r(d, \tilde{x}, \ddot{x})\, dG(\ddot{x} \mid d, x), \quad k = 1, 2, \ldots$$

(ii) If B.1(c.i) and B.2 also hold, then for each (d, x) in supp (D, X)

$$\rho(d, x) = \int r(d, \tilde{x}, \ddot{x}) dG(\ddot{x} \mid x)$$

exists and is finite, and

$$\rho = \mu. \quad ■$$

Covariate-Conditioned average effect of intervention $d \rightarrow d^*$ to $\mathcal{D}$ given $X = x$:

$$\Delta\rho(d, d^*, x) \equiv \rho(d^*, x) - \rho(d, x)$$

By theorem 4.1

$$\Delta\rho(d, d^*, x) = \mu(d^*, x) - \mu(d, x)$$

Under some further technical conditions

$$\frac{\partial\rho}{\partial d_j}(d, x) = \frac{\partial\mu}{\partial d_j}(d, x)$$

Example:

Suppose

$$Y \overset{c}{=} D\beta_0 + \widetilde{X}\gamma_0 + \ddot{X}\delta_0$$

Then

$$E(Y \mid D, X) = D\beta_0 + \widetilde{X}\gamma_0 + E(\ddot{X} \mid D, X)\delta_0$$

Suppose $\ddot{X} \perp D \mid X$. Then

$$E(\ddot{X} \mid D, X) = E(\ddot{X} \mid X)$$

Suppose $E(\ddot{X} \mid X) = X'\alpha_0$

Then

$$E(Y \mid D, X) = D\beta_0 + X'\alpha^*$$

$$X'\alpha^* = \widetilde{X}\gamma_0 + X'\alpha_0\delta_0$$

Theorem 4.1 implies

$$\begin{aligned}\Delta\rho(d,d^*,x) &= d^*\beta_0 + x'\alpha^* \\ &\quad - (d\beta_0 + x'\alpha^*) \\ &= (d^* - d)\beta_0\end{aligned}$$

$$\frac{\partial\rho}{\partial d_j}(d,x) = \beta_{0j}$$

Coefficients α^* have no causal interpretation

Regression $E(Y \mid D, X) = D\beta_0 + X'\alpha^*$ involves

endogenous D
endogenous X
including "irrelevant" $W, \widetilde{B}$

Yields causally meaningful result!

SELECTION OF COVARIATES

Proposition 4.4 Given Assumption B.1(a) suppose $D \overset{c}{=} c(X,U)$, where c is a measurable function and U is a random vector such that $\ddot{X} \perp U \mid X$. Then $\ddot{X} \perp D \mid X$, that is, B.2 holds.■

Corollary 4.5 Suppose Assumptions B.1(a,b,c.i) hold and that $D \overset{c}{=} c(X,U)$, where c is a measurable function and U is a random vector such that $\ddot{X} \perp U \mid X$. If the other assumptions of Theorem 4.1 hold, then its conclusions also hold.
■

We seek X s.t.

$$D \overset{c}{=} c(X,U)$$

$$\ddot{X} \perp U \mid X$$

When D is a response beyond researcher control,

$$D \overset{c}{=} \ddot{c}(\widetilde{X}^*, \ddot{X}^*),$$

for some unknown measurable function $\ddot{c}$ and "D-relevant" explanatory variables $(\widetilde{X}^*, \ddot{X}^*)$, say, where $\widetilde{X}^*$ is observable and $\ddot{X}^*$ is not

$$D \overset{c}{=} c(\widetilde{X}^*, \widetilde{X}^+, \ddot{X}^*) = \ddot{c}(\widetilde{X}^*, \ddot{X}^*)$$

$$U = \ddot{X}^*$$

$$X = (\widetilde{X}^*, \widetilde{X}^+)$$

In Assumption B.1, we represent the covariates as $X = (W, \widetilde{X}, \widetilde{B})$. It follows that the proxy settings W can be constructed as observable responses of the form

$$W \stackrel{c}{=} w(\widetilde{X}^*, \ddot{X}^*, \ddot{X}, \widetilde{X}, \widetilde{B}, \ddot{B}, V),$$

$\ddot{B}$ W-relevant unobservable attributes

V unobservable random variables

$$V \perp D \mid \widetilde{X}^*, \ddot{X}^*, \widetilde{X}, \ddot{X}, \widetilde{B}, \ddot{B}$$

COVARIATE SELECTION GUIDELINES

Include

$\widetilde{X}^*$ D-relevant explanatory variables

$\widetilde{X}$ Y- relevant explanatory variables

$\widetilde{B}$ Attribute proxies for $\ddot{A}, \ddot{A}^*$

W^+Predictive proxies formed as responses W^+ to $\ddot{Z}, \ddot{Z}^*$

Exclude

- Variables preceded by $\mathcal{Y}$
- Variables preceded by $\mathcal{D}$
- Variables not justified by inclusion criteria

Guidelines do not guarantee

$$\ddot{X} \perp D \mid X$$

Some additional assumptions permit a test of

$$\ddot{X} \perp D \mid X$$

that involves only observables

SUMMARY AND CONCLUSIONS

Settable system framework unifies

- Classical Cowles Commission approach – simultaneous systems of structural equations.
- Labor econometrics and related treatment effect approaches
- Machine learning approach to causal analysis.

Additional Benefits

- Generalizes notion of exogeneity (conditional exogeneity)
- Clarifies interpretation of regression coefficients and their estimates
- Relaxes SUTVA of treatment effects literature
- Extends machine learning framework to permit mutual causality
- Provides insight into selection of covariates
- Delivers tests for identification of causal effects
- Provides basis for extending concept of instrumental variables

Theorem 8.1 Given Assumption B.1, let $C \equiv (\tilde{C}, \ddot{C})$, where $\tilde{C}$ is an observable finitely dimensioned attribute vector, and $\ddot{C}$ is an unobservable countably dimensioned attribute vector such that $C \perp D \mid (\ddot{X}, X)$; and let observable responses V be generated as

$$V = v(\ddot{X}, X, C, U),$$

where v is some measurable function, $\ddot{X} \equiv (\ddot{Z}, \ddot{A})$, $X \equiv (W, \tilde{Z}, \tilde{A}, \tilde{B})$, and U is generated by settable variables $\mathcal{U}$ such that

$$U \perp D \mid \ddot{X}, X, C.$$

If Assumption B.2 also holds, that is, $\ddot{X} \perp D \mid X$, then (i) $D \perp (V, \tilde{C}) \mid X$; and (ii) $\ddot{X} \perp D \mid V, X, \tilde{C}$. ■

Theorem 8.1 Given Assumption B.1, let $C \equiv (\tilde{C}, \ddot{C})$, where $\tilde{C}$ is an observable finitely dimensioned attribute vector, and $\ddot{C}$ is an unobservable countably dimensioned attribute vector such that $C \perp D \mid (\ddot{X}, X)$; and let observable responses V be generated as

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where v is some measurable function, $\ddot{X} \equiv (\ddot{Z}, \ddot{A})$, $X \equiv (W, \tilde{Z}, \tilde{A}, \tilde{B})$, and U is generated by settable variables $\mathcal{U}$ such that

$$U \perp D \mid \ddot{X}, X, C.$$

If Assumption B.2 also holds, that is, $\ddot{X} \perp D \mid X$, then (i) $D \perp (V, \tilde{C}) \mid X$; and (ii) $\ddot{X} \perp D \mid V, X, \tilde{C}$. ■