This is the English translation of article published in a journal "Bulletein assciacii "Istorij i computer", Moscow, 1983, N 31. P. 266-269.

Sergey Nefedov
The Russian Academy of Sciences
Institute of history and archaeology
Ural department
E-mail: nsa@etel.ru

The Elementary Model of Population Growth

Many works are devoted to study of population dynamics in an agricultural society with the help of mathematical models (For example**:** Komlos and Artzrouni 1990, Steinmann, Prskawetz and Feichtinger 1998, Kögel and Prskawetz 2001**).** However many models contain unknown coefficients, which influence their behavior. In this brief note we offer the elementary differential model, which has not uncertain parameters.

Let $N(t)$ is a population at the moment t. $K(t)$ is stores of grain after the collecting of crop measured by an amount of minimum annual portions (1 portion is approximately 240 kg of a grain). r is a growth rate of the population in congenial conditions. A square of sowings and a crop depend on a population. They aspire to some constant, when the population grows. We shall consider, that the crop is determined by the formula $P=aN/(N+d)$, where a and d are some constants. We use the usual logistics equation for exposition of population dynamics

(1)

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

K is a current capacity in this equation. This magnitude corresponds to an number $K(t)$ of the accumulated annual portions in our case. N portions is spent for one year, and the accretion of stores will be equal

$$\frac{dK}{dt} = P - N = \frac{aN}{(N+d)} - N$$

(2)

So, we have the elementary system of two differential equations (1) - (2). This system has a position of equilibrium, when the population and the stores remain constants. It is a point $K_0 = N_0 = a-d$.

If we fasten N to 0 in the formula for dP/dN, we shall receive a/d - crop (in an amount of portions), received by one farmer in congenial conditions (when the population is small and he can cultivate a maximum square). Thus, the magnitude $q = a/d$ shows, how much man (including itself) one farmer can feed in congenial conditions (or how much families one peasant family can feed). It is known, that q changes in limits $1.2 < q < 2$ usually. It is convenient to express a and d through q and N_0:

$$d = N_0/(q-1), \qquad a = qN_0/(q-1).$$

N_0 is known (or we can equate it to 1 conditionally). So, in this model we have two constants *r* and *q*, having a real sense and varying in known limits: *0,01 < r < 0,02, 1,2 < q < 2.*
The usual methods of dynamic systems research allow to establish, that the system (1) - (2) has characteristic numbers

$$\lambda_{1,2} = -\frac{r}{2} \pm i\sqrt{-D}, and D = \frac{r^2}{4} - r\left(1 - \frac{1}{q}\right)$$

(Magnitude D < 0 in all range r and q). It means, that the system (1) - (2) generates damped oscillations. The first oscillations can have various period, but when curve comes nearer to position of equilibrium, period is close to

$$T = \frac{2\pi}{\sqrt{-D}}$$

The period T increases, if *r* and *q* decrease, and period T decreases, if *r* and *q* increase.

$q\backslash r$	0.01	0.02
1.2	154	110
2.0	89	63

Tab. 1. The period of oscillations at various r and q (in years).

The first cycle can be much longer than the usual extent of a cycle in a case, when the initial population is small. The presence of large stocks makes a illusion of well-being, but when the stocks are exhausted, the demographic catastrophe comes (fig. 1). The population can decrease in 2-4 times for a short time. The abundance of free lands occurs after the catastrophe, the population grows again, but the excessive growth results in a catastrophe again. The second cycle is closer to the standard period T, and the fall of a population has the smaller sizes. Oscillations gradually decrease in next cycles, and the reduction of a population has no catastrophic character now.

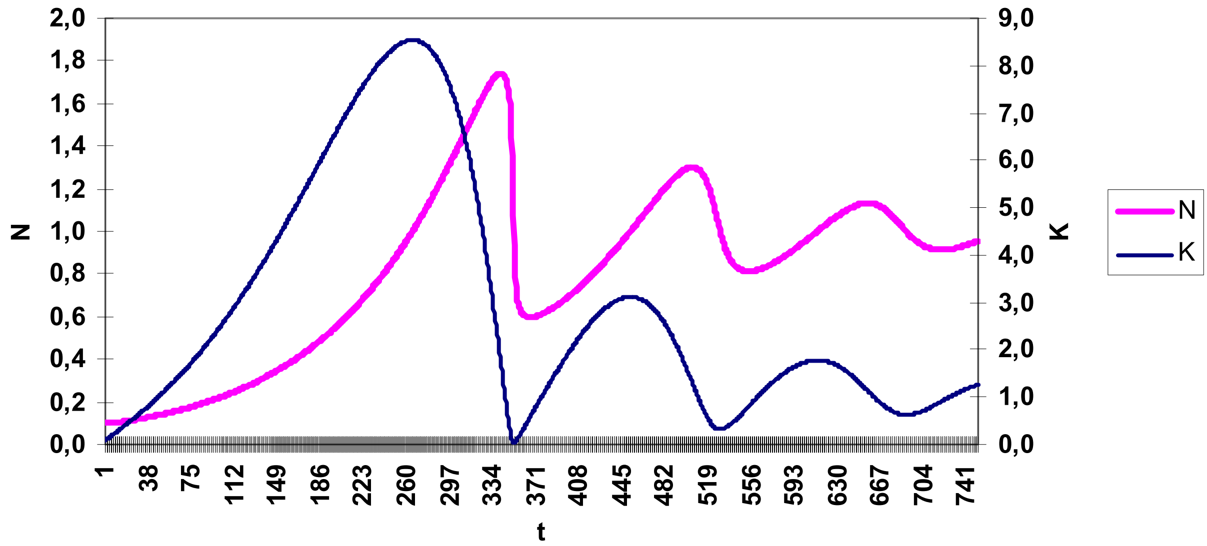

Fig. 1 Example of account on model (r=0,016, p=1,2).

$q\backslash r$	0.01	0.02
1.2	0,46	0,33
2.0	0,64	0,53

Tab. 2. Factor of reduction of amplitude for one cycle (for cycles near to a point of stabilization).

Thus, the dynamics of an agricultural population has oscillatory character.

Near to point of stabilization the period of oscillations has the extent of one-two century, and the amplitude decreases approximately twice for one cycle.

The oscillations damp theoretically, but in practice various casual influences remove the system from a position of equilibrium, and a new series of damped oscillations begins then. Such picture of damped oscillations was received earlier by J. Komlos and S. Nefedov at modeling the growth of the population of Europe in 13th-18th centuries (Komlos and Nefedov 2002).

The author thanks professor Peter Turchin for the useful discussion of work.

References

Kögel, T. and Prskawetz, A. 2001. Agricultural Productivity Growth and Escape from the Malthusian Trap. *Journal of Economic Growth*. 6: 337-57.

Komlos, J. and Artzrouni, M. 1990. Mathematical Investigations of the Escape from the Malthusian Trap. *Mathematical Population Studies*. 2: 269-287.

Komlos, J. and Nefedov S. 2002. A Compact Macromodel of Pre-Industrial Population Growth. *Historical Methods*. 35: (no. 2): 92-94.

Steinmann, G., Prskawetz, A. and Feichtinger, G. 1998. A Model on the Escape from the Malthusian Trap. *Journal of Population Economics.* 11: 535-550.

Turchin, P. 2003. *Historical Dynamics: Why States Rise and Fall.* Princeton University Press.