The Silk Roads. Mathematical model.

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Abstract

The paper concerns the problem of mathematical modeling of historical processes. The dynamics of the Silk Roads is described by means of formal spatial equations. Historical data conveys the facts that the location of the trade routes known as the Silk Roads altered profoundly enough from epoch to epoch. These changes arise from a number of causes – population oscillations, economic trends, diseases and warfare – all these factors affected the dynamics of the Silk Roads and sometimes predetermined its rise and demise. Mathematical simulation of the Silk Roads could help to distinguish the most significant factors and to estimate where and when these factors were especially efficient. In this paper we examine the hypothesis considered by Jeremy Bentley. This attitude implies that one of the most important causes of the Silk Road prosperity was the development of large-scale empires. It promoted the trade greatly. On the one hand big empires stimulate exchange of commodities for the rise of supply and demand of bulk and prestige goods, on the other hand they construct roads and related infrastructure that also induces active trade. Finally they bring stability to vast areas; it is of high importance for negotiations as well. The model takes all these aspects into account and demonstrates the oscillations of the Silk Roads activity induced by the rise and demise of large empires such as Roman, Parthian, Mongol empires, Han and Tang dynasties etc. Simulation gives also some additional curious results.

The subject of historical simulations does not engross wide attention yet, but it seems to be very promising, owing to the actual interest in mathematical applications in social sciences. Obviously there are no exact formal laws or formulae as yet that can describe the behavior (or widely the evolution) of the social medium. However, the experience of exact sciences shows that possible formalization of the knowledge could yield the results more intensional than ever before. It is enticing to operate with some "social equations" and to obtain formal results and forecast for social systems, the same way as we do with physical laws and systems. For the moment we only start this way to formal social science and the attempts of historical simulations could supplement the efforts of social modeling with extensive data and fresh ideas.

Mathematical modeling of social processes is a discipline that emerges naturally due to onrush of mathematical methods and computing machinery firstly and considerable complication of social interactions, risks and threats growth secondly. Great success in modeling of complex physical processes induced attempts of applying mathematical apparatus to the social sciences. However, while the first steps of social modeling have shown that sometimes simple linear equations can give adequate predictions, these steps have also exposed the fundamental problems. One of such problems is a "human factor". Mechanical systems have no free choice, so traditional physicomathematical apparatus does not allow describing complex systems possessing the freedom of choice. How can one describe social systems by means of mathematics then? Fortunately, modern physics (not only physics but also biology, system analysis, cybernetics, synergetics etc.) successfully deal with complex open systems that have no deterministic description and so to say enjoy freedom of choice (Every sufficiently complex system has free choice due to internal instabilities. The effect of “mixing layer” allows a complex system to generate information and so to make choice). So the difficulty of the mathematical analysis of society is an inapplicability of deterministic approach to the social systems. This fact seems to contradict traditional concepts of physical science. The law is considered there as a direct cause-and-effect relation. Oppositely traditional social sciences pay attention firstly to the particularities of the social phenomena. Every attempt of cause-and-effect description is inevitable nullified by numeral exceptions. This is an immanent property of complex systems. Usually there are no obvious pure factors that affect social events. Necessity and determinism is complemented by random fluctuations and plurality of possible life-lines. But this is not a reason to declare indeterminism and impossibility of any kind of laws. Modern physics however already has wide experience of complex systems description and related methods are well enough elaborated. This success makes mathematical methods attractive to economists, sociologists, psychologists, etc. So fortunately specialists both in physical and social sciences have found now common approach to mathematical description of social processes. It is rather difficult to find understanding and common language but the great number of interdisciplinary works denotes the efficiency of such approach.

Mathematical modeling of social processes is widely used first of all for pragmatic forecasting. The most mathematized social discipline is economics. The section of economics of the most interest and attention is the stock exchange dynamics. This is the area of the fastest enrichment or bankruptcy. So the forecast here is pragmatically most demanded item.

The opposite side of the mathematicians’ interest in the modern society daily needs is the following contradiction. Without basic formal theory of society they try to construct models of very complicated processes. It is hardly possible. Modern society is too complicated a system, the result of continuous evolution. It includes both backwashes of the ancient social structures and influence of modern conditions. Evolution changes the society, but the original structure is not replaced, it is only transformed and developed. So without a basic formal theory of less complicated former society we can hardly construct formal theory of modern society. Another problem of the modern society theory is that every theoretical hypothesis must be proved by a successful prediction. For macro-processes the time of perdition also has macro-scale, so the today’s hypothesis can be proved only in years or centuries. So there is no possibility to verify and modify the theory. For small-scale processes, prediction prove demands less time, but the behavior of small-scale social systems usually highly depends on the individual behavior of the humans – members of the system. But here we face again the problem of the human factor, which is still unsolved.

So in order to construct formal theory it seems reasonable to consider macro-systems and to work with pre-modern less complicated societies. This way we can avoid both the problem of long-term prediction (we have information on the “future” dynamics of the pre-modern social systems) and the problem of the human factor (which is not so actual for a large-scale system). Further the construction a theory *from historical simple social systems to modern hierarchical complex systems* will take place. History gives us an extensive set of information, which is practically sufficient both for construction and verification of the formal theory. The historical process is unique and unrepeatable but the same basic processes are observed in different societies in different ages. This denotes that there are some basic laws of social dynamics, and these laws are to be found and formalized.

So the perspectives of historical modeling are very promising. However there still exist few scientific works of this kind (Guseynova, Ustinov, Pavlovskii 1981; Nefedov 2001; Turchin 2002; Malkov 2002). The lack of historical numeric data is one of the most important problems. Formal theory could help to reconstruct some data, but it has not been developed yet. For these reasons it is now important to find the usable research direction, well-behaved process, most described object. Being formalized these newly obtained objective laws could help to derivate new laws and reconstruct historical data.

This paper is a work of this kind. There are two goals – to propose some formal apparatus and to apply it to some historical processes. As it was noted above the attempts of macro-systems description seem to be the most creative approach under the current conditions. There are some brilliant examples of mathematical modeling of the historical dynamics of large-scale societies. (Nefedov 2001; Turchin 2002; Malkov 2002). However, they basically consider the internal relations between internal factors and indices of the society. Actually, the external factors such as geographical properties are not taken into account. They are included into the model but do not play the main part. It is acceptable for large practically isolated agrarian societies, which are considered in these works. However, it is evident that there existed not only self-sufficient agrarian states but also a great number of societies that were essentially dependent on transit trade between agrarian states. Societies of this type obviously depend on geography and location of agrarian neighbors. Consequently the main factor here is a spatial factor.

Thus the main intention of this work is to consider spatial historical dynamics and to propose a model of spatial trade. Being constructed as a common model it will be adapted below for a real historical process.

In order to construct the model it is reasonable to base on related disciplines. For spatial trade model the spatial economics is a basic discipline. The experience of mathematical physics that deal with spatial processes will be also valuable.

A continuous model of transportation was proposed and developed by Martin Beckmann (Beckmann 1952). It concerns the process of commodity transportation in some geographical region (e.g. urban territory). However, it was constructed for trade flow optimization and do not pretend to describe the evolution of real flows and trade routes. So the model must be modified and generalized in order to describe spatial historical dynamics.

Beckmann suggests that some commodity sources (producer) and sinks (consumer) should be located at some points of considered geographical region. Every point of region is also characterized by a trough-passing commodity flow and transportation cost trough the point. The problem considered by Beckmann was to find optimal flows under given distribution of producers and consumers and given distribution of transportation costs. Beckmann proposes his model of the spatial market in assumption that "traders must not suffer losses. This means that the gain form trade exactly equals transportation costs…" (Beckmann, Puu 1985: 16). This model gives the stationary distribution of commodity flows. Beckmann proves that this distribution is optimal as the transpiration costs are strictly minimal along every flow line.

This model is wholesome for the field of spatial economics. It can be used for effective trade flow control. Nevertheless there are some limitations that preclude from applying it to the problems of historical dynamics.

Firstly it is a stationary model and it does not describe the dynamics itself. History is a non-stationary process, evolution, at times slow and at times fast transient. To describe history the model must be dynamical.

Secondly the requirement of optimal route choice is unrealistic under the conditions of the lack of information, while the lack of information is inevitable for real processes. Beckmann assumes that the trader chooses the route of strictly minimal transportation cost. But it is clear that a real trader (especially an ancient one) deals only with rough estimations of transportation costs. The model proposed by Beckmann gives an ambiguous solution for "neutral circuits" (Beckmann, Puu 1985: 38) when between two points of a region two distinct flow paths of equal cost exist. In this case however an infinitesimal variation of the cost along one of these paths destroys the neutral circuit and the trader will unambiguously choose the best path. In other words, micro variations cause dramatic macro changes. This situation is obviously unrealistic. A real trader more likely chooses the route randomly (in a certain sense), but the probability of his choice essentially depends on a roughly estimated transportation costs along the corresponding path. Not only transportation costs but also risks, habits, prestige and other factors affect the choice and they are the more significant the closer the costs of two equivalent paths are.

Finally the use of transportation costs as a territory characteristic works well only under the conditions of the modern society. It is problematic enough to reconstruct such data for ancient ages. So there is a need of transportation costs estimation method. It might be reasonable to introduce into practice another meaningful parameter that could describe the trade conductivity of a territory and could be less dependent on currency and prices, more considering non-monetary factors of choice (risk, prestige etc.) and measurable or estimable at the same time.

Thus Beckmann's model requires modification and generalization in order to be applicable to the problems of spatial historical dynamics.

Let us consider a closed region of spatial one-commodity market. Suggest

T(x,y) is the density of commodity,

q(x,y) is the excess density of production, i.e. the difference between the density of production and the density of consumption (*q* is positive if production exceeds consumption at this point, otherwise *q* is negative),

p(x,y) is the distribution of commodity price

The divergent law for this process:

$$\frac{\partial T}{\partial t} = -\mathrm{div}\mathbf{J} + q$$

where **J** is the commodity flow vector.

This well-know equation describes a continuous flow of any substance (e.g. heat flow, liquid flow etc.) In the given case it can be verbalized as follows:

"The increase $\frac{\partial T}{\partial t}$ of commodity density is the sum of the increase due to production *q* and the increase due to difference $-\mathrm{div}\mathbf{J}$ between the incoming and outgoing flow"

The price dynamics can be linearly considered as

$$\frac{\partial p}{\partial t} = \gamma(D - S)$$

where *D* is the demand density at the point, *S* is the supply density and *γ* is the constant of proportionality that implies the supply-demand disbalance sensitivity of market prices.

This equation can be verbalized as:

"The prices grow if demand exceeds supply and fall if demand is less than supply"

In assumption that there are no commodity selling limitations suggest

$$\frac{\partial T}{\partial t} = S - D$$

That is “Overstocking takes place if supply is greater than demand and there are active sales if demand exceeds supply”

Finally, main assumption of the model is that

$$\mathbf{J} = k \cdot \mathrm{grad} p$$

where commodity conduction coefficient k is the coefficient of proportionality. It will be discussed below.

This equation means that the flow of commodity is proportional to the gradient of the commodity price. The verbalization is the following:

“The flow of commodity transportation between adjacent points is the more intensive the greater the difference of prices at this points is”

This equation is the key difference from Beckmann’s model. Beckmann assumes that the flow must have the same direction as the price gradient (as it is known, the gradient vector is directed along the lines of the quickest ascent – in the considered case the lines of the quickest price rise).

Our assumption (unlike Beckmann’s one) implies that not only the direction of the flow is the same as the gradient direction, but also the absolute value of the flow is proportional to the amount of the gradient.

This refinement looks as a slight modification but it is very essential. Namely, this modification withdraws the problem of “neutral circuits” and the problem of decision making under the conditions of the lack of information. That is the traders can define the amount of the inter-local trade using the local properties of the market. They do not demand the secondhand information about adjacent and remote markets.

One of the advantages of the proposed model is that it does not fundamentally contradict Beckmann’s model. Moreover, Beckmann’s optimal stationary solution can be reached in this dynamical model as the time tends to infinity. That means that the system at whole eventually comes to the stationary optimal solution where the cost of transportation is minimal. It is a more realistic behavior – suppose the system was initially stable but suddenly a considerable change of conditions (new production startup, bankruptcy of an enterprise, armed conflict at the region of the trade route) occurs. Due to the lack of information the flows do not come to the optimal configuration immediately after the change of conditions but as the time passes (if conditions do not change dramatically in the sequel) the flows stabilize and become optimal under established conditions. Note that parameter γ corresponds to the speed of the information propagation and system response. The more the γ value is the faster the optimal steady-state pattern establishes.

Assembling all previous equations we can derive:

$$\frac{\partial p}{\partial t} = \gamma(\mathbf{div}(k \cdot \mathbf{grad}p) - q)$$

This equation is well known. This is a “heat conduction equation” that describes the evolution of the spatial system with heat sources and sinks and with an uneven heat-conduction coefficient. Heat equation is well studied (Tikhonov, Samarskii 1951) theoretically and practically (i.e. there exist strong analytical treatments and computational methods). This experience gives us a wide field for effective application of the equation.

So the commodity-conduction coefficient is mathematically equal to the heat-conduction coefficient. Thus it will be discussed with respect to this analogy. Commodity-conduction coefficient (CCC) is a very important factor for a spatial market. The higher this coefficient is the more profitable the conditions both for producers and consumers are. Low CCC results in low prices for producers and high prices for consumers. The difference between these prices corresponds to transportation expenses. This situation obviously is not favorable and can reduce both production and consumption. CCC is related to Beckmann’s transportation costs coefficient. However CCC is more general. It involves not only economical properties but also non-monetary aspects such as risks, prestige, habits etc. Moreover, CCC is more measurable for historical processes. For example if we have an estimation of the amount of the commodity flow between two enough isolated towns and an estimation of the price at each town – we can define the value of CCC along the route connecting these towns. Certainly it is not so easy as described above because the flow and the prices are not constant at time, but the model is dynamical too so it is possible to distinguish the causes of changes – external conditions influence or evolution of the market itself.

Our next step is to approve the model. Let us apply it to some real historical process. If the results of modeling will be appropriate then we can believe the model is correct. So we need to pick out an example of a spatial market system that is large enough spatially and temporally, well enough described and analyzed. There is a system that perfectly satisfies all these conditions. It is a famous trade-route system known as the Silk Roads.

The Silk Roads is a unique phenomenon. It is the most long-life large scale trade-route system in the world. It was not only a merchant route but also a basic factor of the unification of Afroeurasia (Chase-Dunn, Hall 1997). The Silk Roads is the system of trade routes with complex historical dynamics. There were three main epochs of the Silk Roads history. The intensification of the Silk Roads trade took place at each epoch. At the end of each epoch the trade diminished considerably. These epochs are: the epoch of the ancient Silk Road (II B.C.E. – III C.E.), the epoch of Islam propagation (VI – IX C.E.), the epoch of the Mongol Empire (XI – XIV C.E.). At each epoch the pattern of the main trade routes was different – strictly speaking these changes will be the object of our further attention.

What were the main factors affecting the intensity and location of the routes? As usual for complex systems there are too many factors that can de involved in consideration. However, this plurality is unacceptable when we deal with a mathematical description as we do now. We cannot use huge mathematical constructions and our first step is to reduce the system and to select one or several governing factors. Fortunately, for the Silk Roads this factor was found.

Calculations show that the main factor that predetermined the location of the Silk Roads was the spatial layout of large-scale empires. This point of view is similar to that of Jerry Bentley (1993), who examined cross-cultural links such as the Silk Roads and implied that the large-scale empires were presumably the main factor of the Silk Roads existence and dynamics: "The era of the ancient silk roads – roughly 200 B.C.E. to 400 C.E. – thus figures as the first major period of cross-cultural encounter. The consolidation of large imperial states pacified enough of Eurasia that trading networks could safely link the extreme ends of the landmass", "Beginning about the sixth century, however, a revival of long-distance trade underwrote a second round of intense cross-cultural encounters. The revival of cross-cultural dealings depend again on the foundation of large imperial states…", "The second period did not so much come to end as it blended into a new era – roughly 1000 to 1350 … The distinctive feature of this era … had to do with the remarkable military and political expansion of nomadic peoples, principally Turks and Mongols, who established vast transregional empires and sponsored regular and interactions between peoples…" (Bentley 1993: 26-27).

Unquestionably the empires themselves are not an independent phenomenon. There are many other factors that induce the rise and fall of empires. Nevertheless here we do not take these factors into account. We only derive that the existence of empires predetermines the pattern of commodity-flows. It does not matter for us why they exist and how they appeared – it is a subject for another mathematical model. For this case we propose the general equation of spatial demodynamics:

$$\frac{\partial u}{\partial t} = \mathbf{div}(k \cdot \varepsilon \cdot \mathbf{grad}u - k \cdot u \cdot \mathbf{grad}H) + q$$

This equation describes spatial evolution of population density u. Here k corresponds to the migration-conductivity, H – the spatial utility function, ε – the amount of undirected migration, q – the difference between birth and mortality rates.

Using this equation it could be possible to solve some problems of spatial historical dynamics, in particular the problem of empire formation. However, the given paper does not involve the application of this equation. Further we assume that the bounds of all empires are given exogenously.

The mechanisms of the empires' influence on the trade are also comprehensive. Large empires demand and supply more goods for prestige-goods trade networks, large empires support roads and other infrastructure, large empires bring stability to the areas of commerce, etc. There are many other components of the beneficial influence of large empires, but fortunately most of them can be easily described in terms of the spatial trade model, which was proposed above:

Large-scale empire increases the commodity conductivity of the territory inside its bounds.

This means that the commodity-conduction coefficient of a geographical region increases when this region belongs to an empire (e.g. after being conquered by imperial troops) and it decreases back when the empire losses control in this region (e.g. after the imperial collapse).

The increase of conductivity results in the following consequences. The transportation costs (as it was discussed above) decreases when the conductivity increases. So the expenses of traders also become lower. Imperial interlinks are faster and safer. The roads inside the imperial bounds become more attractive for traders even if a shorter path outside the empire exists.

All these factors can cause changes. The original location of main commodity-flows can become less profitable and therefore instable after the formation of an empire nearby. The general flow pattern can change considerably form epoch to epoch as the empires rise and fall, even if the locations of the main commodity producer and consumer remain constant.

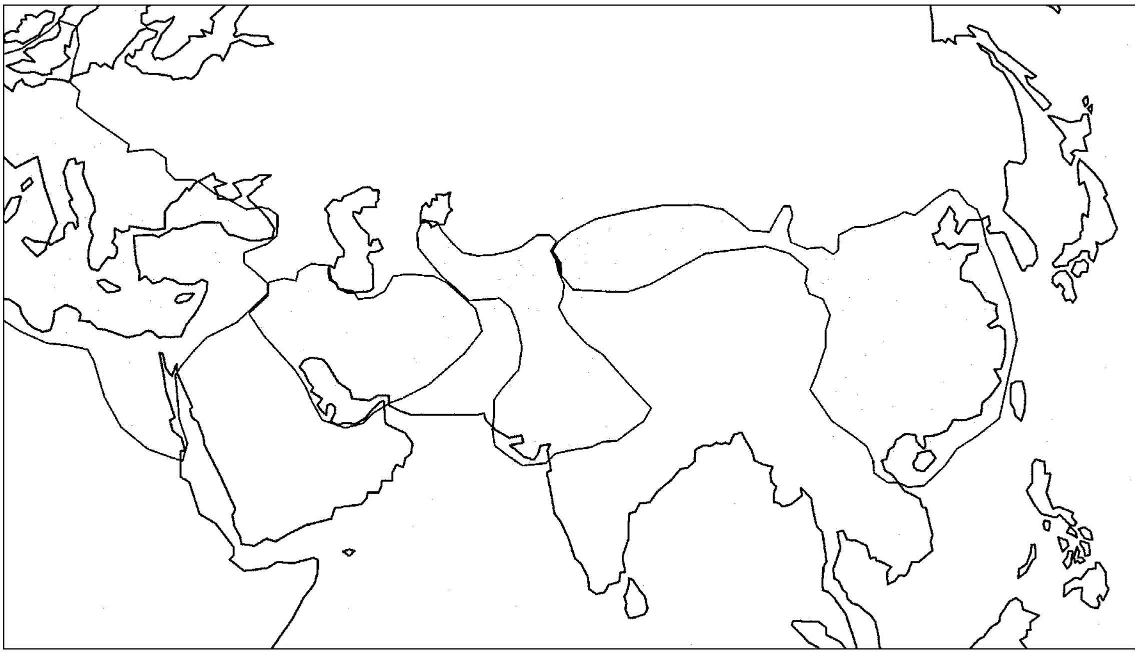
So the simulation of the Silk Roads involves the following assumptions:

1. The model of spatial trade is used as a mathematical basis.
2. Initial conductivity of each geographical point is estimated using the conditions of the respective territory.
3. Three historical epochs are considered – the epoch of the ancient Silk Road, the epoch of Islam, and the epoch of Mongols.
4. For each epoch two points are assigned – the point of the main production of commodity and the point of main consumption of commodity.
5. For each epoch the layout of main empires is assigned – the commodity-conductivity coefficient increases inside the empire at the respective epoch.

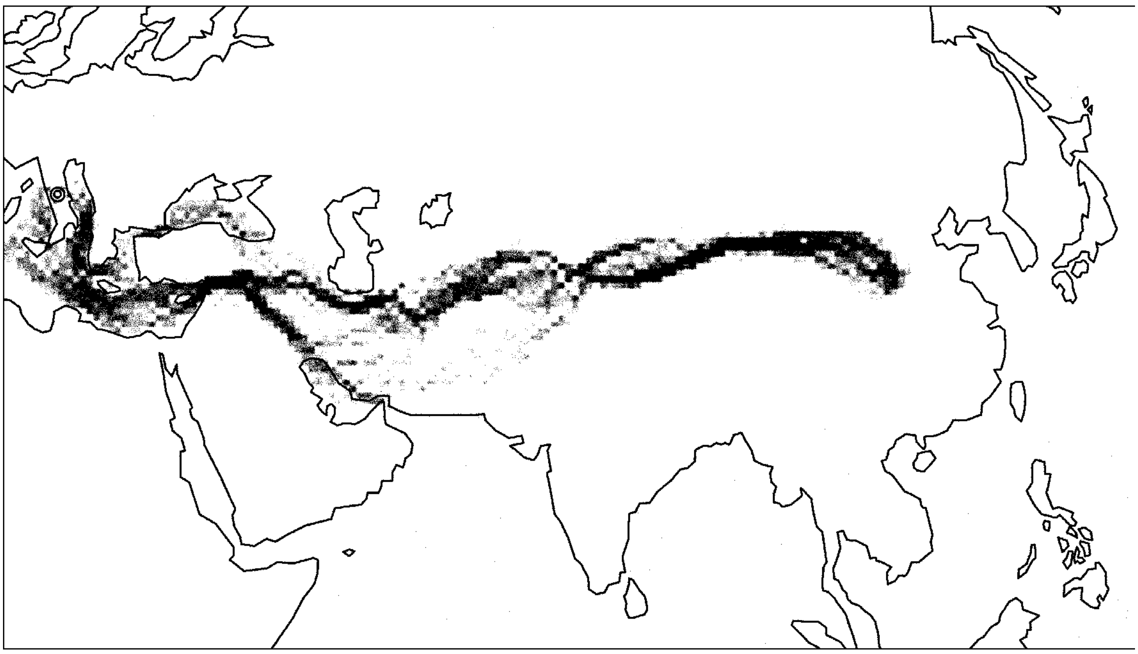
Mathematically the model corresponds to the parabolic equation with a point source, point sink, uneven coefficient and the boundary condition of the formal flow equaling zero. The simulation was counted out using the finite-difference methods.

The results of calculations are given below. First picture illustrates the layout of the main empires of an epoch. Second picture corresponds to the numerical results. Third picture corresponds to the real historical data ('World History', 1956-1958).

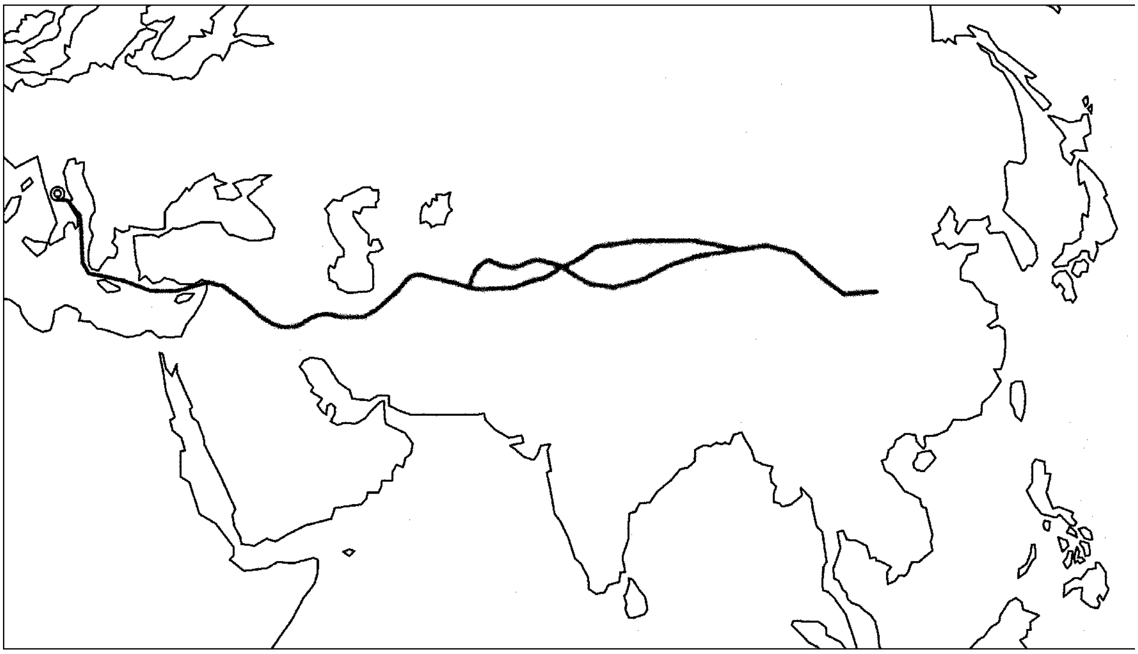
II B.C.E. – III C.E. The epoch of the ancient Silk Roads.



The main empires of the epoch of the Ancient Silk Road were the Roman, Parthian, Kushan and Han Empires. It was the first time in history that Eurasia became an integrated system. However, only prestige-goods networks were integrated. Military networks and bulk goods networks of these world-systems were much smaller than prestige-goods network. (Chase-Dunn, Hall: 2003).

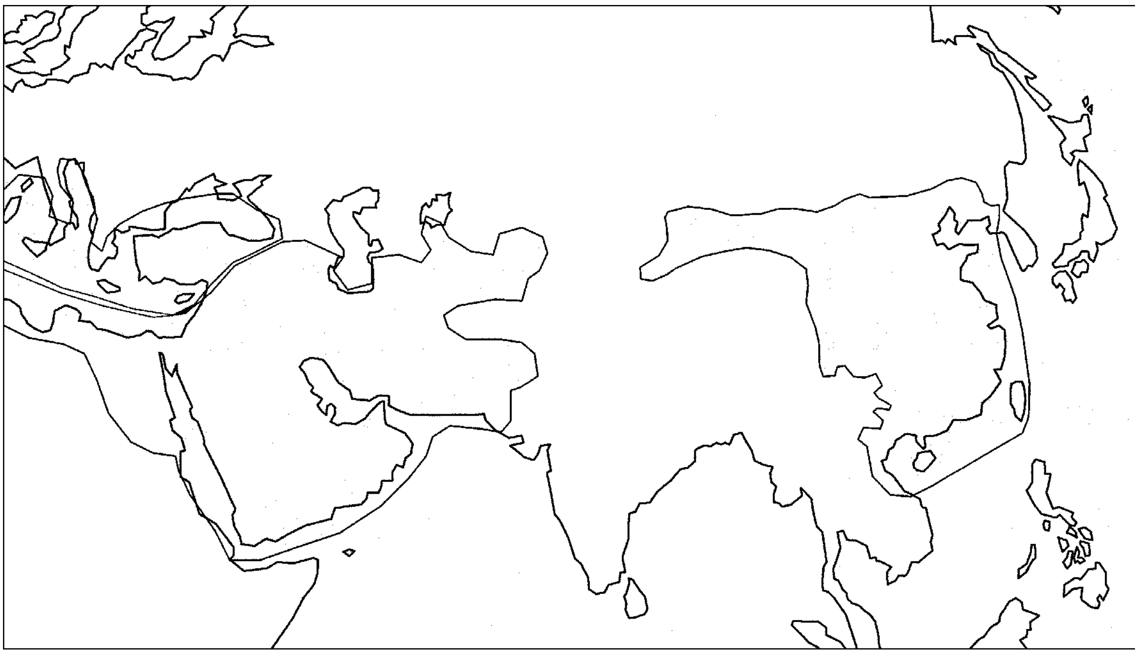


Calculation results for the first epoch. The darker the point is the higher the commodity transportation through the point is.

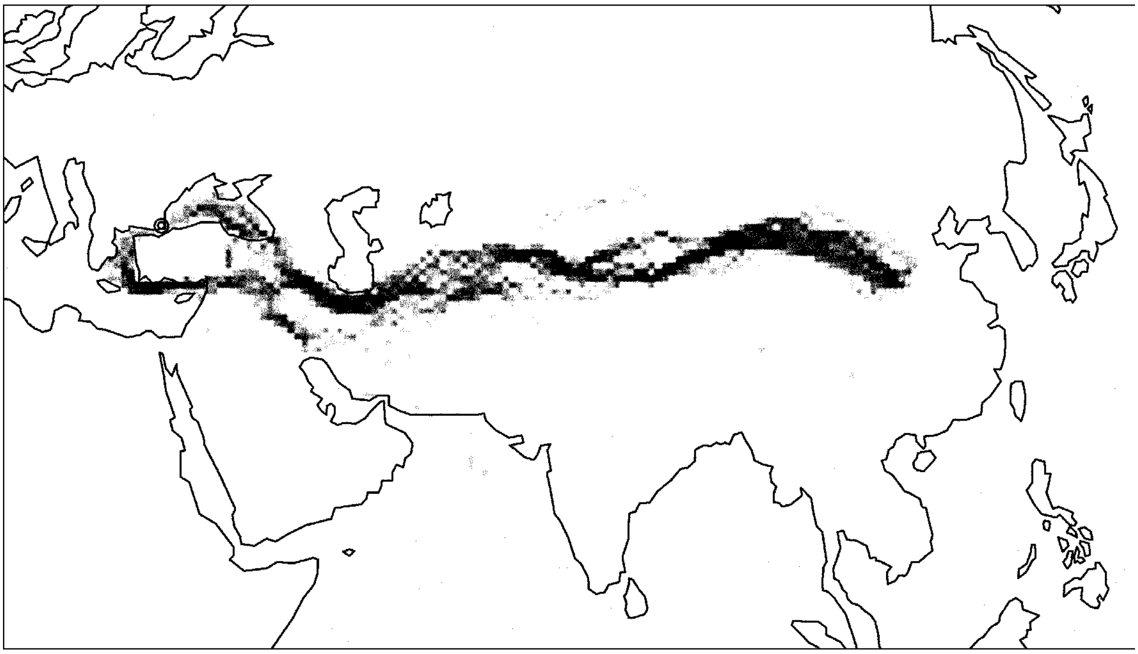


Historical data on the Silk Roads location for the first epoch

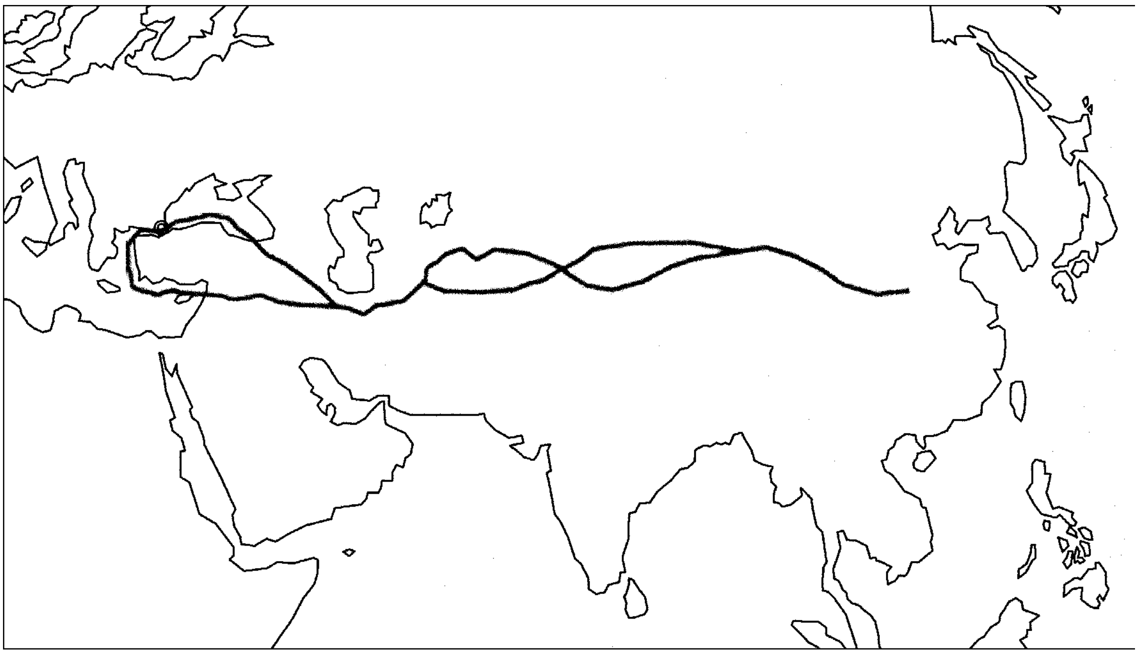
VI – IX C.E. The epoch of Islam propagation



Main empires are the Byzantine Empire, Islam community, Tang Empire.

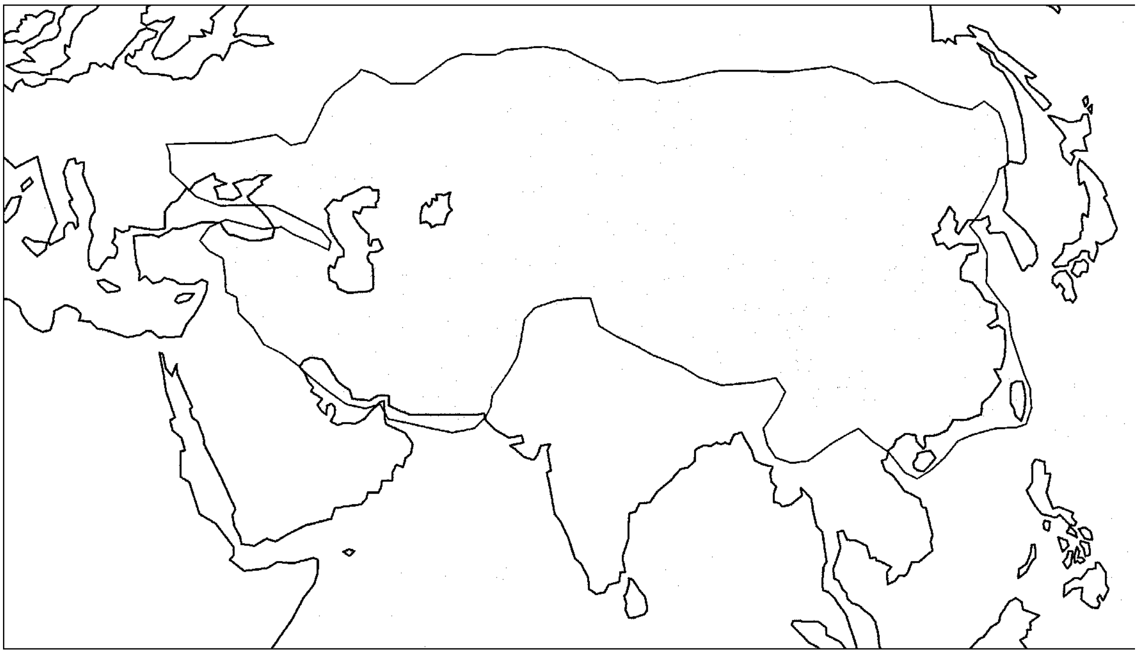


Calculation results.

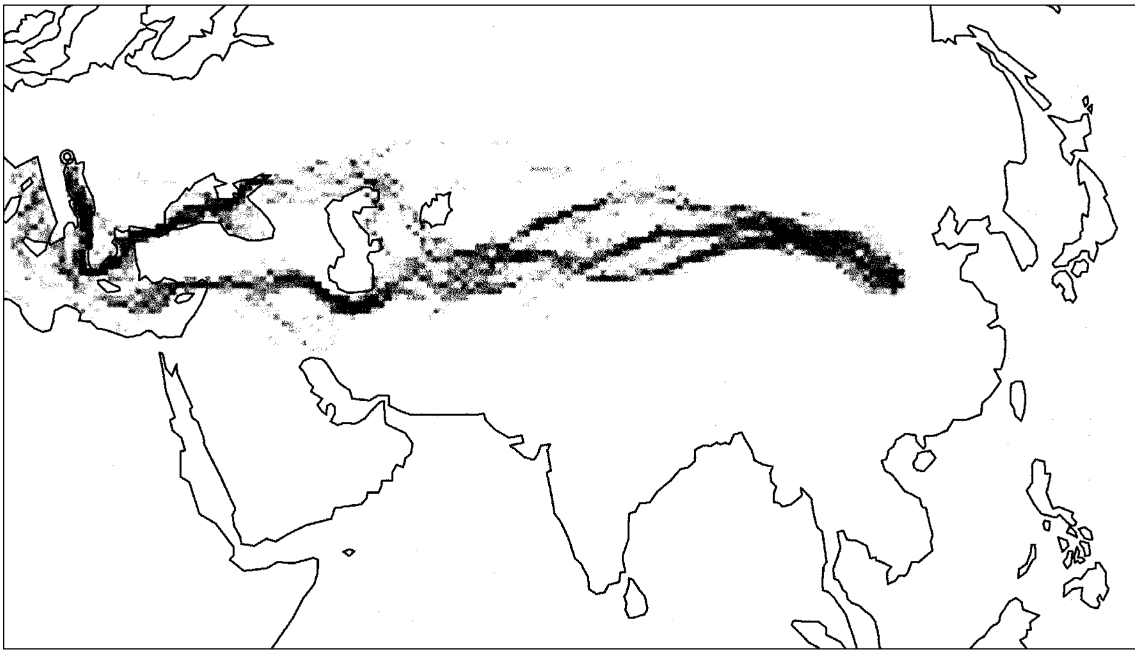


Main routes of The Silk Roads System at this epoch.

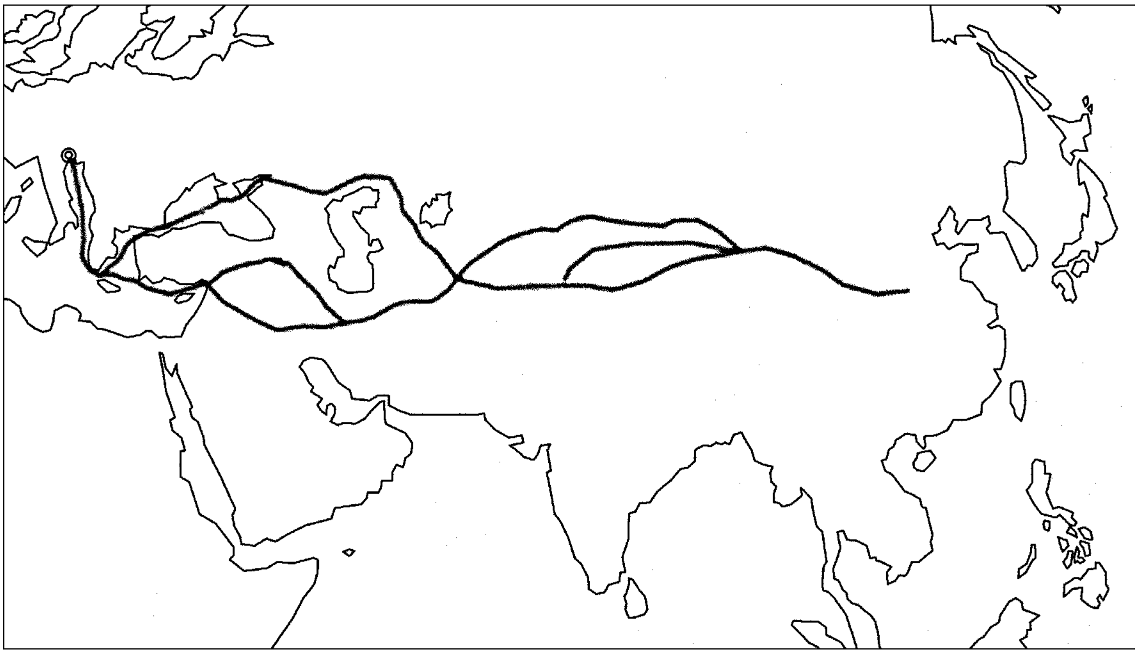
XI – XIV A.D. The epoch of the Mongol Empire



Huge Mongol empire was the main power of this epoch. It was the first time that military networks reached such a scale.



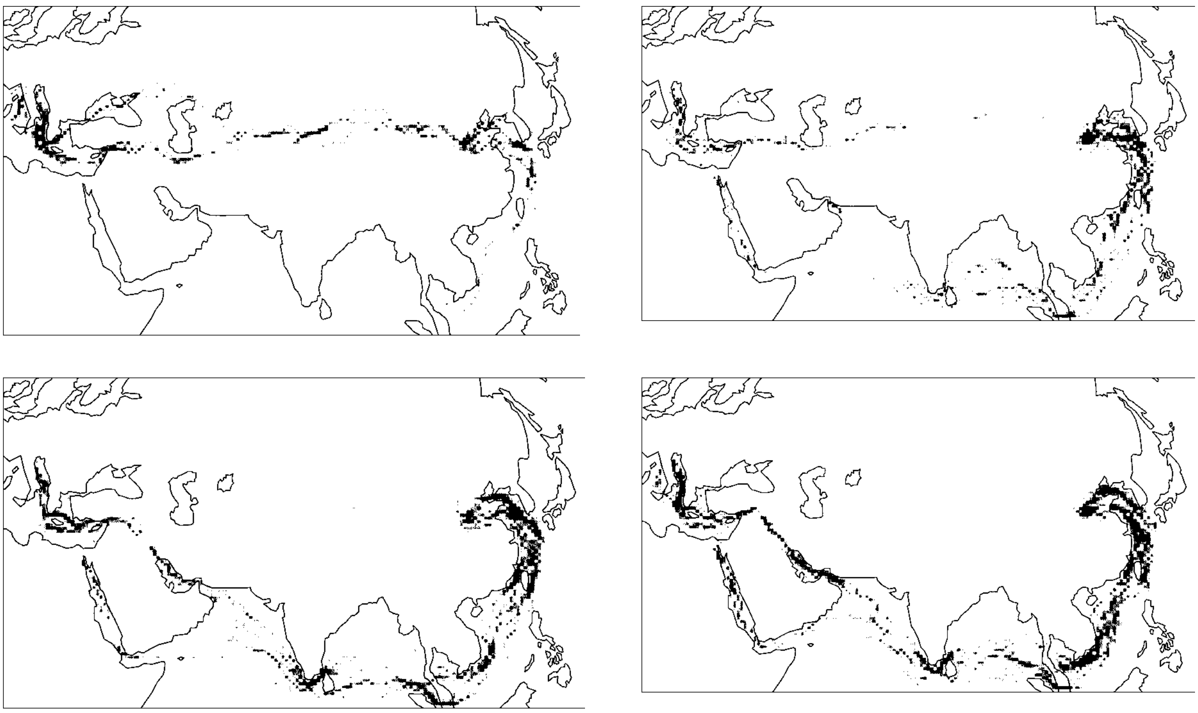
Due to activity of the Mongols a stable intensive Silk Roads route to the north of the Caspian Sea appears for the first time.



The system of main routes with Venice as a destination point in Europe

After the fall of the Mongol Empire the system of the Silk Roads demises and never retakes its former importance. The model gives a curious explanation of this fact. According to the model European galleons were the cause of this process.

If we increase the commodity-conduction of the Indian Ocean we obtain the following sequence of pictures:



This increase of inductivity can be considered (with respect to the Bentley's theory) as a result of naval expansion of new European empires.

Let us say a few words in conclusion. The obtained results look curious, but they are still too rough. We used rather simple a model that involves only one factor – the factor of large empires. It is impossible to expect precise prediction at every point as it is in physics. To adjust the results we must take into account other factors, add new equations and expand the model. However, new equations must be proposed only after successful approbation and testing. So it is not reasonable to hurry with the model expansion. The results show that the model is valid in principle, so our further work is to apply it to more precise historical data, propose more effective methods of conductivity evaluation and try to obtain actual predictions.

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